

HITTING TIMES FOR GAUSSIAN PROCESSES

BY LAURENT DECREUSEFOND AND DAVID NUALART¹

GET/Telecom Paris and University of Kansas

We establish a general formula for the Laplace transform of the hitting times of a Gaussian process. Some consequences are derived, and particular cases like the fractional Brownian motion are discussed.

1. Introduction. Consider a zero mean continuous Gaussian process $(X_t, t \geq 0)$, and for any $a > 0$, we denote by τ_a the hitting time of the level a defined by

$$(1.1) \quad \tau_a = \inf\{t \geq 0 : X_t = a\} = \inf\{t \geq 0 : X_t \geq a\}.$$

Thus, the map $(a \mapsto \tau_a)$ is left-continuous and increasing, hence, with right limits. The map $(a \mapsto \tau_{a+})$ is right continuous where

$$\tau_{a+} = \lim_{b \downarrow a, b > a} \tau_b = \inf\{t \geq 0 : X_t > a\}.$$

Little is known about the distribution of τ_a . It is explicitly known in particular cases like the Brownian motion. If X is a fractional Brownian motion with Hurst parameter H , there is a result by Molchan [5] which stands that

$$P(\tau_a > t) = t^{-(1-H)+o(1)}$$

as t goes to infinity.

When X is a standard Brownian motion, it is well known that

$$(1.2) \quad E(\exp(-\alpha\tau_a)) = \exp(-a\sqrt{2\alpha})$$

for all $\alpha > 0$. This result is easily proved using the exponential martingale

$$M_t = \exp(\lambda B_t - \frac{1}{2}\lambda^2 t).$$

Received September 2006; revised January 2007.

¹Supported in part by the NSF Grant DMS-06-04207.

AMS 2000 subject classifications. Primary 60H05; secondary 60G15, 60H07.

Key words and phrases. Fractional Brownian motion, hitting times.

This is an electronic reprint of the original article published by the Institute of Mathematical Statistics in *The Annals of Probability*, 2008, Vol. 36, No. 1, 319–330. This reprint differs from the original in pagination and typographic detail.

By Doob's optional stopping theorem applied at time $t \wedge \tau_a$ and letting $t \rightarrow \infty$, one gets $1 = E(M_{\tau_a}) = E(\exp(\lambda B_{\tau_a} - \lambda^2 \tau_a / 2))$. Since $B_{\tau_a} = a$, we thus obtain (1.2). If we consider a general Gaussian process X_t , the exponential process

$$M_t = \exp(\lambda X_t - \frac{1}{2} \lambda^2 V_t),$$

where $V_t = E(X_t^2)$ is no longer a martingale. However, it is equal to 1 plus a divergence integral in the sense of Malliavin calculus. The aim of this paper is to take advantage of this fact in order to derive a formula for $E(\exp(-\frac{1}{2} \lambda^2 V_{\tau_a}))$. We derive an equation involving this expectation in Theorem 3.4, under rather general conditions on the covariance of the process. As a consequence, we show that if the partial derivative of the covariance is nonnegative, then $E(\exp(-\frac{1}{2} \lambda^2 V_{\tau_a})) \leq 1$, which implies that V_{τ_a} has infinite moments of order p for all $p \geq \frac{1}{2}$ and finite negative moments of all orders. In particular, for the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, we have the inequality

$$E(\exp -\alpha \tau_a^{2H}) \leq \exp(-a\sqrt{2\alpha})$$

for all $\alpha, a > 0$.

The paper is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus, and the main results are proved in Section 3.

2. Preliminaries on Malliavin calculus. Let $(X_t, t \geq 0)$ be a zero mean Gaussian process such that $X_0 = 0$ and with covariance function

$$R(s, t) = E(X_t X_s).$$

We denote by \mathcal{E} the set of step functions on $[0, +\infty)$. Let \mathcal{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R(t, s).$$

The mapping $\mathbf{1}_{[0,t]} \rightarrow X_t$ can be extended to an isometry between \mathcal{H} and the Gaussian space $H_1(X)$ associated with X . We will denote this isometry by $\varphi \rightarrow X(\varphi)$.

Let \mathcal{S} be the set of smooth and cylindrical random variables of the form

$$(2.1) \quad F = f(X(\phi_1), \dots, X(\phi_n)),$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$.

The *derivative operator* D of a smooth and cylindrical random variable F of the form (2.1) is defined as the \mathcal{H} -valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

The derivative operator D is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. The Sobolev space $\mathbb{D}^{1,2}$ is the closure of \mathcal{S} with respect to the norm

$$\|F\|_{1,2}^2 = E(F^2) + E(\|DF\|_{\mathcal{H}}^2).$$

The *divergence operator* δ is the adjoint of the derivative operator. We say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by $\text{Dom } \delta$, if

$$|E(\langle DF, u \rangle_{\mathcal{H}})| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$. In this case $\delta(u)$ is defined by the duality relationship

$$(2.2) \quad E(F\delta(u)) = E(\langle DF, u \rangle_{\mathcal{H}}),$$

for any $F \in \mathbb{D}^{1,2}$.

Set $V_t = R(t, t)$. For any $\lambda > 0$, we define

$$M_t = \exp(\lambda X_t - \frac{1}{2}\lambda^2 V_t).$$

Formally, the Itô formula for the divergence integral, proved, for instance, in [1], implies that

$$(2.3) \quad M_t = 1 + \lambda \delta(M \mathbf{1}_{[0,t]}),$$

where $M \mathbf{1}_{[0,t]}$ represents the process $(s \mapsto M_s \mathbf{1}_{[0,t]}(s), s \geq 0)$. However, the process $M \mathbf{1}_{[0,t]}$ does not belong, in general, to the domain of the divergence operator. This happens, for instance, in the following basic example.

EXAMPLE 1. Fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $(B_t^H, t \geq 0)$ with the covariance

$$(2.4) \quad R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

In this case, the processes $(B_s^H \mathbf{1}_{[0,t]}(s), s \geq 0)$ and $(\exp(\lambda B_s^H - \frac{1}{2}\lambda^2 s^{2H}) \mathbf{1}_{[0,t]}(s), s \geq 0)$ do not belong to $L^2(\Omega; \mathcal{H})$ if $H \leq \frac{1}{4}$ (see [2]).

In order to define the divergence of $M \mathbf{1}_{[0,t]}$ and to establish formula (2.3), we introduce the following additional property on the covariance function of the process X .

(H0) The covariance function $R(t, s)$ is continuous, the partial derivative $\frac{\partial R}{\partial s}(s, t)$ exists in the region $\{0 < s, t, s \neq t\}$, and for all $T > 0$,

$$\sup_{t \in [0, T]} \int_0^T \left| \frac{\partial R}{\partial s}(s, t) \right| ds < \infty.$$

Notice that this property is satisfied by the covariance (2.4) for all $H \in (0, 1)$.

Define

$$(2.5) \quad \delta_t M = \frac{1}{\lambda}(M_t - 1).$$

The following proposition asserts that $\delta_t M$ satisfies an integration by parts formula, and in this sense, it coincides with an extension of the divergence of $M\mathbf{1}_{[0,t]}$.

PROPOSITION 2.1. *Suppose that (H0) holds. Then, for any $t > 0$, and for any smooth and cylindrical random variable of the form $F = f(X_{t_1}, \dots, X_{t_n})$, we have*

$$(2.6) \quad E(F\delta_t M) = E\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \int_0^t M_s \frac{\partial R}{\partial s}(s, t_i) ds\right).$$

PROOF. First notice that condition (H0) implies that the right-hand side of equation (2.6) is well defined. Then, it suffices to show equation (2.6) for a function of the form

$$f(x_1, \dots, x_n) = \exp\left(\sum_{i=1}^n \lambda_i x_i\right),$$

where $\lambda_i \in \mathbb{R}$. In this case we have, for all $0 < t_1 < \dots < t_n$,

$$\begin{aligned} & \frac{1}{\lambda} E(F(M_t - 1)) \\ &= \frac{1}{\lambda} \exp\left\{\frac{1}{2} \sum_{i=1}^n \lambda_i \lambda_j R(t_i, t_j)\right\} \left(\exp\left\{\sum_{i=1}^n \lambda \lambda_i R(t, t_i)\right\} - 1\right) \\ &= \sum_{i=1}^n \int_0^t \exp\left\{\frac{1}{2} \sum_{i=1}^n \lambda_i \lambda_j R(t_i, t_j) + \lambda \sum_{i=1}^n \lambda_i R(s, t_i)\right\} \lambda_i \frac{\partial R}{\partial s}(s, t_i) ds \\ &= \int_0^t E\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) M_s \frac{\partial R}{\partial s}(s, t_i)\right) ds, \end{aligned}$$

which completes the proof of the proposition. \square

In many cases like in Example 1 with $H > \frac{1}{4}$, the process $M\mathbf{1}_{[0,t]}$ belongs to the space $L^2(\Omega; \mathcal{H})$, and then the right-hand side of equation (2.6) equals

$$E\langle DF, M\mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}.$$

In this situation, taking into account the duality formula (2.2), equation (2.6) says that $M\mathbf{1}_{[0,t]}$ belongs to the domain of the divergence and $\delta(M\mathbf{1}_{[0,t]}) = \delta_t M$.

3. Hitting times. In this section we will assume the following conditions:

- (H1) The partial derivative $\frac{\partial R}{\partial s}(s, t)$ exists and it is continuous in $[0, +\infty)^2$.
- (H2) $\limsup_{t \rightarrow \infty} X_t = +\infty$ almost surely.
- (H3) For any $0 \leq s < t$, we have $E(|X_t - X_s|^2) > 0$.

Under these conditions, the process X has a continuous version because

$$\begin{aligned} E(|X_t - X_s|^2) &= R(t, t) + R(s, s) - 2R(s, t) \\ &= \int_s^t \left[\frac{\partial R}{\partial u}(u, t) - \frac{\partial R}{\partial u}(u, s) \right] du \\ &\leq 2|t - s| \sup_{s \leq u \leq t} \left| \frac{\partial R}{\partial u}(u, t) \right|. \end{aligned}$$

For any $a > 0$, we define the hitting time τ_a by (1.1). We know that $P(\tau_a < \infty) = 1$ by condition (H2). Set

$$(3.1) \quad S_t = \sup_{s \in [0, t]} X_s.$$

From the results of [6], it follows that, for all $t > 0$, the random variable S_t belongs to the space $\mathbb{D}^{1,2}$. Furthermore, condition (H3) allows us to compute the derivative of this random variable.

LEMMA 3.1. *For all $t > 0$, with probability one, the maximum of the process X in the interval $[0, t]$ is attained in a unique point, that is, $\tau_{S_t} = \tau_{S_t}^+$ and $DS_t = \mathbf{1}_{[0, \tau_{S_t}]}$.*

PROOF. The fact that the maximum is attained in a unique point follows from condition (H3) and Lemma 2.6 in Kim and Pollard [4]. The formula for the derivative of S_t follows easily by an approximation argument. \square

We need the following regularization of the stopping time τ_a . Suppose that φ is a nonnegative smooth function with compact support in $(0, +\infty)$ and define for any $T > 0$

$$(3.2) \quad Y = \int_0^\infty \varphi(a)(\tau_a \wedge T) da.$$

The next result states the differentiability of the random variable Y in the sense of Malliavin calculus and provides an explicit formula for its derivative.

LEMMA 3.2. *The random variable Y defined in (3.2) belongs to the space $\mathbb{D}^{1,2}$, and*

$$(3.3) \quad D_r Y = - \int_0^{S_T} \varphi(y) \mathbf{1}_{[0, \tau_y]}(r) d\tau_y.$$

PROOF. Clearly, Y is bounded. On the other hand, for any $r > 0$, we have

$$\{\tau_a > r\} = \{S_r < a\}.$$

Therefore, we can write using Fubini's theorem

$$Y = \int_0^\infty \varphi(a) \left(\int_0^{\tau_a \wedge T} d\theta \right) da = \int_0^T \left(\int_{S_\theta}^\infty \varphi(a) da \right) d\theta,$$

which implies that $Y \in \mathbb{D}^{1,2}$ because $S_\theta \in \mathbb{D}^{1,2}$, and

$$D_r Y = - \int_0^T \varphi(S_\theta) D_r S_\theta d\theta = - \int_0^T \varphi(S_\theta) \mathbf{1}_{[0, \tau_{S_\theta}]}(r) d\theta.$$

Finally, making the change of variable $S_\theta = y$ yields

$$D_r Y = - \int_0^{S_T} \varphi(y) \mathbf{1}_{[0, \tau_y]}(r) d\tau_y. \quad \square$$

Notice that $M_Y = \exp(\lambda X_Y - \frac{1}{2} \lambda^2 V_Y)$. Hence, letting $t = Y$ in equation (2.5) and taking the mathematical expectation of both members of the equality yields

$$(3.4) \quad E(M_Y) = 1 + \lambda E(\delta_t M|_{t=Y}).$$

We are going to show the following result which provides a formula for the left-hand side of equation (3.4).

LEMMA 3.3. *Assume conditions (H1), (H2) and (H3). Then, we have*

$$(3.5) \quad E(M_Y) = 1 - \lambda E \left(M_Y \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(Y, \tau_y) d\tau_y \right).$$

PROOF. The proof will be done in two steps.

Step 1. We claim that for any function $p(x)$ in $\mathcal{C}_0^\infty(\mathbb{R})$ we have

$$(3.6) \quad E(\delta_t M p(Y)) = -E \left(\int_0^t M_s p'(Y) \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) d\tau_y ds \right).$$

We can write $Y = \int_0^T \psi(S_\theta) d\theta$, where $\psi(x) = \int_x^\infty \varphi(a) da$. Consider an increasing sequence D_n of finite subsets of $[0, T]$ such that their union is dense in $[0, T]$. Set $Y_n = \int_0^T \psi(S_\theta^n) d\theta$, and $S_\theta^n = \max\{X_t, t \in D_n \cap [0, \theta]\}$. Then, Y_n is a Lipschitz function of $\{X_t, t \in D_n\}$. Hence, formula (2.6), which holds for Lipschitz functions, implies that

$$E(\delta_t M p(Y_n)) = -E \left(p'(Y_n) \int_0^T \varphi(S_\theta^n) \left(\int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_\theta^n}) ds \right) d\theta \right).$$

The function $r \rightarrow \int_0^t M_s \frac{\partial R}{\partial s}(s, r) ds$ is continuous and bounded by condition (H1). As a consequence, we can take the limit of the above expression as n tends to infinity and we get

$$E(\delta_t M p(Y)) = -E\left(p'(Y) \int_0^T \varphi(S_\theta) \left(\int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_\theta}) ds \right) d\theta\right).$$

Finally, making the change of variable $S_\theta = y$ yields (3.6).

Step 2. We write

$$E(\delta_t M|_{t=Y}) = E\left(\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \delta_t M p_\varepsilon(Y - t) dt\right),$$

where $p_\varepsilon(x)$ is an approximation of the identity, and by convention, we assume that $\delta_t M = 0$ if t is negative. We can commute the expectation with the above limit by the dominated convergence theorem because

$$\begin{aligned} \int_{-\infty}^{\infty} |\delta_t M| p_\varepsilon(Y - t) dt &= \int_{-\infty}^{\infty} \frac{1}{\lambda} |M_t - 1| p_\varepsilon(Y - t) dt \\ &\leq \frac{1}{\lambda} \sup_{0 \leq t \leq T+1} (|M_t| + 1), \end{aligned}$$

if the support of $p_\varepsilon(x)$ is included in $[-\varepsilon, \varepsilon]$, and $\varepsilon \leq 1$. Hence,

$$(3.7) \quad E(\delta_t M|_{t=Y}) = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} E(\delta_t M p_\varepsilon(Y - t)) dt.$$

Using formula (3.6) yields

$$\begin{aligned} (3.8) \quad &E(\delta_t M p_\varepsilon(Y - t)) \\ &= - \int_0^t E\left(p'_\varepsilon(Y - t) M_s \left(\int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) d\tau_y \right)\right) ds. \end{aligned}$$

Hence, substituting (3.8) into (3.7) and integrating by parts, we obtain

$$\begin{aligned} &E(\delta_t M|_{t=Y}) \\ &= - \lim_{\varepsilon \rightarrow 0} E\left(\int_{-\infty}^{\infty} p'_\varepsilon(Y - t) \left(\int_0^t M_s \left(\int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) d\tau_y \right) ds \right) dt\right) \\ &= - \lim_{\varepsilon \rightarrow 0} E\left(\int_{-\infty}^{\infty} p_\varepsilon(Y - t) \left(M_t \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial t}(t, \tau_y) d\tau_y \right) dt\right). \end{aligned}$$

Notice that

$$\left| \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(s, \tau_y) d\tau_y \right| \leq T \sup_{0 \leq s, u \leq T} \left| \frac{\partial R}{\partial s}(s, u) \right| \|\varphi\|_\infty.$$

Hence, applying the dominated convergence theorem, we get

$$\begin{aligned}
E(M_Y) &= 1 + \lambda E(\delta_t M|_{t=Y}) \\
&= 1 - \lambda \lim_{\varepsilon \rightarrow 0} E\left(\int_{-\infty}^{\infty} p_{\varepsilon}(Y-t) \left(M_t \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(t, \tau_y) d\tau_y\right) dt\right) \\
&= 1 - \lambda E\left(M_Y \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(Y, \tau_y) d\tau_y\right). \quad \square
\end{aligned}$$

The next step will be to replace the function $\varphi(x)$ by an approximation of the identity and let T tend to infinity. Notice that (3.5) still holds for $\varphi(x) = \mathbf{1}_{[0,b]}(x)$ for any $b \geq 0$. In this way we can establish the following result.

THEOREM 3.4. *Assume conditions (H1), (H2) and (H3). For any $a > 0$ and $\lambda \in \mathbb{R}$, we have*

$$\begin{aligned}
&\int_0^a E(M_{\tau_y}) dy \\
(3.9) \quad &= a - \lambda E\left(\int_0^a \int_0^1 M_{z\tau_{y^+} + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_{y^+} + (1-z)\tau_y, \tau_y) dz d\tau_y\right).
\end{aligned}$$

Notice that we are not able to differentiate with respect to a , the integral in the rightmost expectation of (3.9), because the (random) measure $d\tau_y$, in general, is not absolutely continuous with respect to the Lebesgue measure.

PROOF OF THEOREM 3.4. Fix $a > 0$. We first replace the function $\varphi(x)$ by an approximation of the identity of the form $\varphi_{\varepsilon}(x) = \varepsilon^{-1} \mathbf{1}_{[0,1]}(x/\varepsilon)$ in formula (3.5). We will make use of the following notation:

$$Y_{\varepsilon,a} = \int_0^{\infty} \varphi_{\varepsilon}(x-a)(\tau_x \wedge T) dx.$$

At the same time we fix a nonnegative smooth function $\psi(x)$ with compact support such that $\int_{\mathbb{R}} \psi(a) da = c$ and we set

$$\begin{aligned}
&\int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}}) \psi(a) da \\
&= c - \lambda \int_{\mathbb{R}} E\left(M_{Y_{\varepsilon,a}} \int_0^{S_T} \varphi_{\varepsilon}(y-a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) d\tau_y\right) \psi(a) da.
\end{aligned}$$

The increasing property of the function $x \rightarrow \tau_x$ implies that $\tau_{a^+} \wedge T \leq Y_{\varepsilon,a} \leq \tau_{a+\varepsilon} \wedge T$. Hence, Y_{ε} converges to $\tau_{a^+} \wedge T$ as ε tends to zero. Thus, almost surely, we have

$$\lim_{\varepsilon \rightarrow 0} M_{Y_{\varepsilon,a}} = \exp(\lambda X_{\tau_{a^+} \wedge T} - \frac{1}{2} \lambda^2 V_{\tau_{a^+} \wedge T}).$$

By the dominated convergence theorem,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} E(M_{Y_{\varepsilon,a}}) \psi(a) da = \int_{\mathbb{R}} E(\exp(\lambda X_{\tau_{a+} \wedge T} - \frac{1}{2} \lambda^2 V_{\tau_{a+} \wedge T})) \psi(a) da.$$

Now, set $F(t) = M_t \frac{\partial R}{\partial s}(t, \tau_y)$. Then, assuming that $\varphi_\varepsilon(x) = \varepsilon^{-1} \mathbf{1}_{[0,1]}(x/\varepsilon)$, we have

$$\begin{aligned} & \int_{y-\varepsilon}^y \varphi_\varepsilon(y-a) M_{Y_{\varepsilon,a}} \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) da \\ &= \frac{1}{\varepsilon^2} \int_{y-\varepsilon}^y \mathbf{1}_{[0,1]} \left(\frac{y-a}{\varepsilon} \right) F \left(\int_a^{a+\varepsilon} \mathbf{1}_{[0,1]} \left(\frac{x-a}{\varepsilon} \right) (\tau_x \wedge T) dx \right) \psi(a) da \\ &= \int_0^1 F \left(\int_0^1 (\tau_{y+\varepsilon\xi-\varepsilon\eta} \wedge T) d\xi \right) \psi(y-\varepsilon\eta) d\eta \\ &= \int_0^1 F \left(\int_0^\eta (\tau_{y+\varepsilon\xi-\varepsilon\eta} \wedge T) d\xi + \int_\eta^1 (\tau_{y+\varepsilon\xi-\varepsilon\eta} \wedge T) d\xi \right) \psi(y-\varepsilon\eta) d\eta. \end{aligned}$$

As ε tends to zero, this expression clearly converges to

$$\psi(y) \int_0^1 F(\eta(\tau_y \wedge T) + (1-\eta)(\tau_{y^+} \wedge T)) d\eta.$$

So, we have proved that

$$\begin{aligned} (3.10) \quad & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} M_{Y_{\varepsilon,a}} \varphi_\varepsilon(y-a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) da \\ &= \psi(y) \int_0^1 M_{z\tau_{y^+} + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_{y^+} + (1-z)\tau_y, \tau_y) dz. \end{aligned}$$

In order to complete the proof of the theorem, we will apply the dominated convergence theorem. We have the following estimate for $y \leq S_T$:

$$\left| \int_{\mathbb{R}} M_{Y_{\varepsilon,a}} \varphi_\varepsilon(y-a) \frac{\partial R}{\partial s}(Y_{\varepsilon,a}, \tau_y) \psi(a) da \right| \leq \|\psi\|_\infty \sup_{s,t \leq T} \left| \frac{\partial R}{\partial s}(s,t) \right| \sup_{t \leq T} |M_t|,$$

which allows us to commute the limit (3.10) with the integral with respect to the measure $P \times d\tau_y$ on the set $\{(\omega, y) : y \leq S_T(\omega)\}$. In this way we get

$$\begin{aligned} & \int_{\mathbb{R}} E(M_{\tau_y}) \psi(y) dy \\ &= \int_{\mathbb{R}} \psi(y) dy \\ & \quad - \lambda E \left(\int_0^{S_T} \psi(y) \int_0^1 M_{z\tau_{y^+} + (1-z)\tau_y} \frac{\partial R}{\partial s}(z\tau_{y^+} + (1-z)\tau_y, \tau_y) dz d\tau_y \right). \end{aligned}$$

Approximating $\mathbf{1}_{[0,a]}$ by a sequence of smooth functions $(\psi_n, n \geq 1)$ and letting T tend to infinity completes the proof. \square

If we assume that the partial derivative $\frac{\partial R}{\partial t}(t, s)$ is nonnegative, then we can derive the following result.

PROPOSITION 3.5. *Assume that X satisfies hypotheses (H1), (H2) and (H3). If $\frac{\partial R}{\partial s}(s, t) \geq 0$, then, for all $\alpha, a > 0$, we have*

$$(3.11) \quad E(\exp(-\alpha V_{\tau_a})) \leq e^{-a\sqrt{2\alpha}}.$$

PROOF. Since $\frac{\partial R}{\partial t}(t, s) \geq 0$, we obtain

$$E(M_{\tau_a}) \leq 1,$$

that is,

$$E(\exp(\lambda a - \frac{1}{2}\lambda^2 V_{\tau_a})) \leq 1,$$

or

$$E(\exp(-\alpha V_{\tau_a})) \leq e^{-a\sqrt{2\alpha}}.$$

The result follows. \square

The above proposition means that the Laplace transform of the random variable V_{τ_a} is dominated by the Laplace transform of τ_a , where τ_a is the hitting time of the level a for the ordinary Brownian motion. This domination implies some consequences on the moments of V_{τ_a} . In fact, for any $r > 0$, we have, multiplying (3.11) by α^r ,

$$(3.12) \quad \begin{aligned} E(V_{\tau_a}^{-r}) &= \frac{1}{\Gamma(r)} \int_0^\infty E(e^{-\alpha V_{\tau_a}}) \alpha^{r-1} d\alpha \\ &\leq \frac{1}{\Gamma(r)} \int_0^\infty e^{-a\sqrt{2\alpha}} \alpha^{r-1} d\alpha \\ &= \frac{2^r \Gamma(r+1/2)}{\sqrt{\pi}} a^{-2r}. \end{aligned}$$

On the other hand, for $0 < r < 1$,

$$(3.13) \quad \begin{aligned} E(V_{\tau_a}^r) &= \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - E(e^{-\alpha V_{\tau_a}})) \alpha^{-r-1} d\alpha \\ &\geq \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - e^{-a\sqrt{2\alpha}}) \alpha^{-r-1} d\alpha. \end{aligned}$$

In particular, for $r \in [1/2, 1)$, $E(V_{\tau_a}^r) = +\infty$.

REMARK 3.6. If X is the standard Brownian motion, its covariance $s \wedge t$ does not satisfy condition (H1), but we still can apply our approach.

It is known from [3] that $d\tau_a$ has no absolutely continuous part and that $\{a, \tau_a = \tau_a^+\}$ is a Cantor set, hence, of zero Lebesgue measure. It follows from this observation and from (3.10) that

$$\int E(M_{\tau_y})\psi(y) dy = \int \psi(y) dy.$$

Choosing $\psi = \mathbf{1}_{[0,a]}$ yields to the expected result:

$$E\left(\int_0^a e^{\lambda y - (\lambda^2/2)V(\tau_y)} dy\right) = a.$$

If X has independent increments and satisfies (H3), then

$$E(e^{-(\lambda^2/2)V(\tau_a)}) = e^{-\lambda a}.$$

This follows easily from the fact that X can be written as a time-changed Brownian motion.

REMARK 3.7. Consider that X is a fractional Brownian motion of Hurst index $H = 1$. Then $R(s, t) = st$, and consequently, $X_t = Yt$, where Y is a one-dimensional standard Gaussian random variable. Then, $\tau_a = \tau_{a^+} = a/Y^+$. It is then easy to compute the Laplace transform of τ_a and we obtain

$$(3.14) \quad E(\exp(-\alpha\tau_a^2)) = \frac{1}{2}e^{-a\sqrt{2\alpha}}.$$

We show now that our formula also yields to the right answer. We just note that $(y \mapsto \tau_y)$ is continuous. This entails that

$$\frac{\partial R}{\partial s}(z\tau_{y^+} + (1-z)\tau_y, \tau_y) = \frac{\partial R}{\partial s}(\tau_y, \tau_y) = \frac{1}{2}V'(\tau_y)$$

and

$$(3.15) \quad \begin{aligned} & \int_0^a E\left(\exp\left(\lambda y - \frac{\lambda^2}{2}V(\tau_y)\right)\right) dy \\ &= a - \frac{\lambda}{2}E\left(\int_0^a \exp\left(\lambda y - \frac{\lambda^2}{2}V(\tau_y)\right)V'(\tau_y) d\tau_y\right). \end{aligned}$$

Set

$$\Psi(a, \lambda) = E\left(\exp\left(\lambda a - \frac{\lambda^2}{2}V(\tau_a)\right)\right),$$

then

$$(3.16) \quad \frac{\partial \Psi}{\partial a}(a, \lambda) = \lambda\Psi(a, \lambda) - \frac{\lambda^2}{2}E\left(M_{\tau_a}\frac{\partial V(\tau_a)}{\partial a}\right).$$

Substitute (3.15) into (3.16) to obtain

$$\frac{\partial \Psi}{\partial a} = 2\lambda \Psi - \lambda.$$

Then, there exists a function $C(\lambda)$ such that

$$\Psi(a, \lambda) = \frac{1}{2} + C(\lambda)e^{2\lambda a} \quad \text{so that } E\left(\exp\left(-\frac{\lambda^2}{2}\tau_a^2\right)\right) = \frac{1}{2}e^{-\lambda a} + C(\lambda)e^{\lambda a}.$$

By dominated convergence, it is clear that, for any λ ,

$$E\left(\exp\left(-\frac{\lambda^2}{2}\tau_a^2\right)\right) \xrightarrow{a \rightarrow \infty} 0,$$

thus, $C(\lambda) = 0$ and

$$E\left(\exp\left(-\frac{\lambda^2}{2}\tau_a^2\right)\right) = \frac{1}{2}e^{-\lambda a}.$$

Changing $\lambda^2/2$ into α gives (3.14).

REMARK 3.8. Consider the case of a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Conditions (H1), (H2) and (H3) are satisfied and we obtain

$$\begin{aligned} & \int_0^a E(M_{\tau_y}) dy \\ &= a - \lambda H E\left(\int_0^a \int_0^1 M_{z\tau_{y+} + (1-z)\tau_y} ([z\tau_{y+} + (1-z)\tau_y]^{2H-1} \right. \\ & \quad \left. - |z(\tau_{y+} - \tau_y)|^{2H-1}) dz d\tau_y\right). \end{aligned}$$

Moreover, $E(e^{-\alpha\tau_a^{2H}}) \leq e^{-a\sqrt{2\alpha}}$, and therefore, $E(\tau_a^p) < \infty$ if $p < H$. According to (3.13), $E(\tau_a^p)$ is infinite if $pH > 1/4$ and (3.12) entails that τ_a has finite negative moments of all orders.

Acknowledgments. This work was carried out during a stay of Laurent Decreusefond at Kansas University, Lawrence, KS. He would like to thank KU for warm hospitality and generous support.

REFERENCES

- [1] ALÒS, E., MAZET, O. and NUALART, D. (2001). Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29** 766–801. [MR1849177](#)
- [2] CHERIDITIO, P. and NUALART, D. (2005). Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$. *Ann. Inst. H. Poincaré Probab. Statist.* **41** 1049–1081. [MR2172209](#)

- [3] ITÔ, K. and MCKEAN, H. P., JR (1974). *Diffusion Processes and Their Sample Paths*. Springer, Berlin. [MR0345224](#)
- [4] KIM, J. and POLLARD, D. (1990). Cube root asymptotics. *Ann. Probab.* **18** 191–219. [MR1041391](#)
- [5] MOLCHAN, G. M. (2000). On the maximum of fractional Brownian motion. *Theory Probab. Appl.* **44** 97–102. [MR1751192](#)
- [6] NUALART, D. and VIVES, J. (1988). Continuité absolue de la loi du maximum d'un processus continu. *C. R. Acad. Sci. Paris Sér. I Math.* **307** 349–354. [MR0958796](#)

DEPARTMENT OF MATHEMATICS OF INFORMATION
COMMUNICATIONS AND CALCULUS
TELECOM PARIS
46, RUE BARRAULT
75634 PARIS
FRANCE
E-MAIL: Laurent.Decreusefond@enst.fr

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
405 SNOW HALL
1460 JAYHAWK BLVD
LAWRENCE, KANSAS 66045-7523
USA
E-MAIL: nualart@math.ku.edu